LOCAL AND GLOBAL WELL-POSEDNESS TO MAGNETO-ELASTICITY SYSTEM

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ABSTRACT. In this article we employ classical tricks to give local and global well-posedness to Magneto-Elasticity System. Different from many cases, we consider the equation which the magnetic field satisfies is Landau-Lifshitz system without viscidity, i.e. the Schrödinger flow. As is well known, people can not obtain global existence of Schrödinger flow at general cases. However, the reason why we do what others can not do is the Schrödinger flow with non-zero convection term.

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CONTENTS

1. INTRODUCTION

The discovery of magnetoelasticity dates back at least to the 19th century. This literature in the area is refered to [1]. Magnetoelastic interaction describes a class of phenomena on the interaction between elastic and magnetic effects: if a ferromagnetic rod is subject to a magnetizing field, the rod changes not only its magnetization but also its length, and in the opposite way, if the rod experiences tension, its length as well as its magnetization changes.

Usually, the evolution of magnetoelasticity is modeled by :

$$
\left\{
\begin{array}{l}
\partial_t v + \nabla_v v + \nabla p + \nabla \cdot (2A\nabla M \odot \nabla M - W'(F)F^\top) - \nu \Delta v = \mu_0 (\nabla H_{ext})^\top M, \\
\nabla \cdot v = 0, \\
\partial_t F + \nabla_v F - \nabla v F = \kappa \Delta F, \\
\partial_t M + \nabla_v M = -\gamma M \times (2A\Delta M + \mu_0 H_{ext}) - \lambda M \times [M \times (2A\Delta M + \mu_0 H_{ext})],
\end{array}
\right. \tag{1.1}
$$

where $\nu \geq 0$ is the viscosity of the fluid, $v : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is velocity, $M : \mathbb{R} \times \mathbb{R}^d \to \mathbb{S}^2$ stands for magnetization and $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ represents the deformation gradient with respect to the velocity v. κ, A and $\mu_0 > 0$ are parameters coming from the Helmholtz free energy. $\gamma > 0$ is the electron gyromagnetic ratio and $\lambda > 0$ is a phenomenological damping parameter. $W : \mathbb{R}^{d\times d} \to \mathbb{R}$ is the elastic energy and "$\top$" means the transpose of a square matrix, while "$\times$" and "$\cdot$" are respectively the vector production of $\mathbb{R}^3$ and the inner production of $\mathbb{R}^d$. The known function $p : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is the hydrodynamic pressure. $\nabla M \odot \nabla M$ is a squre matrix of order d whose the $(i,j)_{1\leq i\leq d, 1\leq j\leq d}$ entry is

$$
(\nabla M \odot \nabla M)_{ij} = \sum_{k=1}^{d} \partial_i M_k \partial_j M_k.
$$

$H_{ext} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ denotes the given external magnetic field. Note that $\nabla_v M$ **is called convection term**.

The system we consider is the following simplified one

$$
\left\{
\begin{array}{l}
\partial_t v + v \cdot \nabla v + \nabla p + \nabla \cdot (\nabla M \odot \nabla M - FF^\top) = \nu \Delta v \\
\nabla \cdot v = 0 \\
\partial_t F + v \cdot \nabla F = \nabla v F \\
\partial_t M + v \cdot \nabla M + M \times \Delta M = 0 \\
|M| = 1,
\end{array}
\right. \tag{1.2}
$$

Moreover, the initial data is imposed on

$$
v(0,x) = v_0(x) \in \mathbb{R}^d, \quad F(0,x) = F_0(x) \in \mathbb{R}^{d\times d}, \quad M(0,x) = M_0(x) \in \mathbb{S}^2
$$

with compatibilities $\nabla \cdot v_0(x) = 0$ and $\det F_0(x) = 1$.

Before describing our main theorems, we firstly give the definition of weak solutions.

Definition 1.1. *We say that*

$$
M \in L^\infty([0,T) \times \mathbb{R}^d) \bigcap C^0([0,T), L^2) \bigcap L^2([0,T), L^2) \quad \textit{with} \quad \nabla M \in L^2([0,T), L^2 \bigcap L^4)
$$

$$
F \in L^2([0,T), L^4 \bigcap L^2) \bigcap C^0([0,T), L^2)
$$

$$
v \in C^0([0,T), L^2) \bigcap L^2([0,T), H^1 \bigcap L^2) \bigcap L^2([0,T), L^4)
$$

is a weak solution of the first type to (1.2) *if, for any set of functions* $(\Phi,\Psi,\chi)\in C^1([0,T),H^1)$ *and* $f\in H^1$*, one has*

$$\begin{cases} \int\langle v(t,\cdot),\Phi(t,\cdot)\rangle - \int\langle v_0,\Phi(0,\cdot)\rangle - \int_0^t\int\langle v,\partial_t\Phi\rangle - \int_0^t\int\langle v, v\cdot\nabla\Phi\rangle + \int_0^t\int\langle\nabla p,\Phi\rangle \\ -\int_0^t\int\langle\nabla M\odot\nabla M - FF^\top,\nabla\Phi\rangle + \nu\int_0^t\int\langle\nabla v,\nabla\Phi\rangle = 0 \\ \int\langle v(t,\cdot),\nabla f\rangle = 0 \\ \int\langle F(t,\cdot),\Psi(t,\cdot)\rangle - \int\langle F_0,\Psi(0,\cdot)\rangle - \int_0^t\int\langle F,\partial_t\Psi\rangle - \int_0^t\int\langle F, v\cdot\nabla\Psi\rangle - \int_0^t\int\langle\nabla vF,\Psi\rangle = 0 \\ \int\langle\Psi(t,\cdot),F(t,\cdot)\rangle - \int\langle\Psi(0,\cdot),F_0\rangle - \int_0^t\int\langle\partial_t\Psi,F\rangle - \int_0^t\int\langle v\cdot\nabla\Psi,F\rangle - \int_0^t\int\langle\Psi,\nabla vF\rangle = 0 \\ \int\langle M(t,\cdot),\chi(t,\cdot)\rangle - \int\langle M_0,\chi(0,\cdot)\rangle - \int_0^t\int\langle M,\partial_t\chi\rangle - \int_0^t\int\langle M,v\cdot\nabla\chi\rangle - \int_0^t\int\langle M\times\nabla M,\nabla\chi\rangle = 0 \\ |M| = 1 \qquad a.e. \end{cases}$$

for any $t\in[0,T)$.

Definition 1.2. *We say that*

$$M\in L^\infty([0,T)\times\mathbb{R}^d)\bigcap C^0([0,T),L^2)\bigcap L^2([0,T),L^2) \quad with \quad \nabla M\in L^2([0,T),L^2\bigcap L^4)$$

$$F\in L^2([0,T),L^6) \quad F^{-1}\in L^\infty([0,T)\times\mathbb{R}^d) \quad G:=F^{-1}-I\in L^2([0,T),L^2\bigcap H^1)\bigcap C^0([0,T),L^2)$$

$$v\in C^0([0,T),L^2)\bigcap L^2([0,T),H^1)\bigcap L^2([0,T),L^4\bigcap L^2)$$

is a weak solution of the second type to (1.2) *if, for any set of functions* $(\Phi,\Psi,\chi)\in C^1([0,T),H^1)$ *and* $f\in H^1$*, one has*

$$\begin{cases} \int\langle v(t,\cdot),\Phi(t,\cdot)\rangle - \int\langle v_0,\Phi(0,\cdot)\rangle - \int_0^t\int\langle v,\partial_t\Phi\rangle - \int_0^t\int\langle v,v\cdot\nabla\Phi\rangle + \int_0^t\int\langle\nabla p,\Phi\rangle \\ -\int_0^t\int\langle\nabla M\odot\nabla M,\nabla\Phi\rangle + \int_0^t\int\langle(F\sharp\nabla G^\top)FF^\top + [(FF^\top)\sharp\nabla G^\top]F^\top,\Phi\rangle + \nu\int_0^t\int\langle\nabla v,\nabla\Phi\rangle = 0 \\ \int\langle v(t,\cdot),\nabla f\rangle = 0 \\ \int\langle G(t,\cdot),\Psi(t,\cdot)\rangle - \int\langle G_0,\Psi(0,\cdot)\rangle - \int_0^t\int\langle G,\partial_t\Psi\rangle - \int_0^t\int\langle G,v\cdot\nabla\Psi\rangle + \int_0^t\int\langle(G+I)\nabla v,\Psi\rangle = 0 \\ \int\langle\Psi(t,\cdot),G(t,\cdot)\rangle - \int\langle\Psi(0,\cdot),G_0\rangle - \int_0^t\int\langle\partial_t\Psi,G\rangle - \int_0^t\int\langle v\cdot\nabla\Psi,G\rangle + \int_0^t\int\langle\Psi,(G+I)\nabla v\rangle = 0 \\ \int\langle M(t,\cdot),\chi(t,\cdot)\rangle - \int\langle M_0,\chi(0,\cdot)\rangle - \int_0^t\int\langle M,\partial_t\chi\rangle - \int_0^t\int\langle M,v\cdot\nabla\chi\rangle - \int_0^t\int\langle M\times\nabla M,\nabla\chi\rangle = 0 \\ |M| = 1, \qquad a.e. \end{cases}$$

Our results are the following

Theorem 1.3. *(Local well-posedness)* $s\geq 2$ *is a given integer and* $d\in\{2,3\}$. $(v_0(x),F_0(x),M_0(x))\in\mathbb{R}^d\times\mathbb{R}^{d\times d}\times\mathbb{S}^2$ *is the initial data satisfing* $v_0,F_0,\nabla M_0\in H^s$*. Moreover, we also assume*

$$\mathcal{E}_0 := \|v_0\|_{H^s}^2 + \|F_0\|_{H^s}^2 + \|\nabla M_0\|_{H^s}^2 < \infty.$$

If $s\geq 3$ *or* $s=2$ *and* $d=2$*, then there exists a positive number* T*, depending only on* $\mathcal{E}_0$*, and a set of functions* (v,F,M) *obeying*

$$v\in L^\infty([0,T],H^s)\bigcap L^2\left([0,T],H^{s+1}\right), \quad F\in L^\infty([0,T],H^s), \quad \nabla M\in L^\infty([0,T],H^s)$$

and

$$\sup_{t\in[0,T]}\left(\|v(t)\|_{H^s}^2 + \|F(t)\|_{H^s}^2 + \|\nabla M(t)\|_{H^s}^2\right) + \int_0^T \nu\|\nabla v\|_{H^s}^2 \lesssim 1,$$

such that (v,F,M) *is a unique local strong solutions to* (1.2)*. If* $s=2$ *and* $d=3$*, there exists at least a local weak solution meeting the above inequality.*

Theorem 1.4. *(Global well-posedness) $s \geq 2$ is a given integers and $d \in \{2,3\}$. $(v_0(x), F_0(x), M_0(x)) \in \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{S}^2$ is the initial data satisfing $v_0, F_0, \nabla M_0 \in H^s$. Moreover, we also assume $G_0 := F_0^{-1} - I$ satisfies*

$$\partial_i G_0^{jk} = \partial_k G_0^{ji} \qquad \forall i,j,k = 1,2,\cdots,d$$

and that

$$detF_0 = 1, \qquad \|v_0\|_{H^s}^2 + \|F_0 - I\|_{H^s}^2 + \|\nabla M_0\|_{H^s}^2 \leq \epsilon_0$$

for some small positive number ϵ_0. If $s \geq 3$ or $s = 2$ and $d = 2$, then there exists a set of functions (v, F, M) with $G := F^{-1} - I$ obeying

$$v \in L^\infty(\mathbb{R}^+, H^s), \quad \nabla v \in L^2(\mathbb{R}^+, H^s), \quad G \in L^\infty(\mathbb{R}^+, H^s), \quad \nabla M \in L^\infty(\mathbb{R}^+, H^s),$$

$$\partial_t v \in L^\infty\left(\mathbb{R}^+, H^{s-2}\right), \qquad \partial_t G \in L^\infty\left(\mathbb{R}^+, H^{s-2}\right)$$

and

$$\|v(t)\|_{H^s}^2 + \|G(t)\|_{H^s}^2 + \|\nabla M(t)\|_{H^s}^2 + \|\partial_t v(t)\|_{H^{s-2}}^2 + \|\partial_t G(t)\|_{H^{s-2}}^2 + \int_0^t \nu \|\nabla v(\tau)\|_{H^s}^2 d\tau \lesssim \epsilon_0$$

for all $t \geq 0$, such that (v, F, M) is a unique global strong solutions to (1.2). If $s = 2$ and $d = 3$, there exists at least a global weak solution meeting the above inequality.

Remark 1.5. *The fact that F is invertible implies that v never vanish and so does the convection term.*

2. Uniqueness of strong solutions

2.1. **Uniqueness of local existence.** Suppose that $(v_i, F_i, M_i)(i \in \{1,2\})$ are two solutions to (1.2).

$$\left\{\begin{array}{ll} \partial_t(v_1 - v_2) & = (v_2 - v_1)\cdot\nabla v_2 + v_1 \cdot \nabla(v_2 - v_1) + \nu\Delta(v_1 - v_2) \\ & +\nabla\cdot[\nabla(M_2 - M_1)\odot\nabla M_2] + \nabla\cdot[\nabla M_1 \odot \nabla(M_2 - M_1)] \\ & +\nabla\cdot[(F_1 - F_2)F_1^\top + F_2(F_1^\top - F_2^\top)] \\ \nabla\cdot(v_1 - v_2) & = 0 \\ \partial_t(F_1 - F_2) & = (v_2 - v_1)\cdot\nabla F_2 + v_1\cdot\nabla(F_2 - F_1) + \nabla(v_1 - v_2)F_1 + \nabla v_2(F_1 - F_2) \\ \partial_t(M_1 - M_2) & = (v_2 - v_1)\cdot\nabla M_1 + v_2\cdot\nabla(M_2 - M_1) + (M_2 - M_1)\times\Delta M_1 \\ & +M_2 \times \Delta(M_2 - M_1). \end{array}\right. \tag{2.1}$$

Define the energy as follow

$$\mathcal{E}(t) := \|v_1(t) - v_2(t)\|_{H^1}^2 + \|F_1(t) - F_2(t)\|_{L^2}^2 + \|M_1(t) - M_2(t)\|_{H^1}^2.$$

It is easy to get

$$\begin{aligned} \frac{d\mathcal{E}}{dt} = & \ 2\int\langle v_1 - v_2, \partial_t v_1 - \partial_t v_2\rangle + 2\int\langle F_1 - F_2, \partial_t F_1 - \partial_t F_2\rangle \\ & +2\int\langle M_1 - M_2, \partial_t M_1 - \partial_t M_2\rangle - 2\int\langle\Delta v_1 - \Delta v_2, \partial_t v_1 - \partial_t v_2\rangle \\ & -2\int\langle\Delta M_1 - \Delta M_2, \partial_t M_1 - \partial_t M_2\rangle \end{aligned} \tag{2.2}$$

Substituting (2.1) into (3.3) gives

$$
\begin{aligned}
\frac{d\mathcal{E}}{dt} \lesssim\ & \int |v_1-v_2|^2\cdot|\nabla v_2| + \int |\nabla M_1-\nabla M_2|\cdot(|\nabla M_1|+|\nabla M_2|)\cdot|\nabla v_1-\nabla v_2| \\
& + \int |\nabla v_1-\nabla v_2|\cdot|F_1-F_2|\cdot(|F_1|+|F_2|) + \int |F_1-F_2|\cdot|v_1-v_2|\cdot|\nabla F_2| \\
& + \int |v_1-v_2|\cdot|v_1|\cdot|\nabla v_1-\nabla v_2| + \int |F_1-F_2|\cdot|\nabla F_1-\nabla F_2|\cdot|v_1| \\
& + \int |\nabla v_1-\nabla v_2|\cdot|F_1|\cdot|F_1-F_2| + \int |F_1-F_2|^2\cdot|\nabla v_2| + \int (|\nabla v_2|+|\nabla v_1|)\cdot|\nabla v_1-\nabla v_2|^2 \\
& + \int |M_1-M_2|\cdot|v_1-v_2|\cdot|\nabla M_1| + \int |M_1-M_2|\cdot|\nabla M_2|\cdot|\nabla M_1-\nabla M_2| \\
& + \int |\nabla v_1-\nabla v_2|\cdot[|\nabla^3 M_1-\nabla^3 M_2|\cdot(|\nabla M_2|+|\nabla M_1|)+|\nabla^2 M_1-\nabla^2 M_2|\cdot\big(|\nabla^2 M_2|+|\nabla^2 M_1|\big)] \\
& + \int |\nabla v_1-\nabla v_2|\cdot[|\nabla^2 F_1-\nabla^2 F_2|\cdot(|F_1|+|F_2|)+|\nabla F_1-\nabla F_2|(|\nabla F_1|+|\nabla F_2|)] \\
& + \int |\nabla v_1-\nabla v_2|\cdot|F_1-F_2|(|\nabla^2 F_1|+|\nabla^2 F_2|) + \int |v_1-v_2|\cdot|\nabla M_1|\cdot|\nabla^2 M_1-\nabla^2 M_2| \\
& + \int |v_2|\cdot|\nabla M_1-\nabla M_2|\cdot|\nabla^2 M_1-\nabla^2 M_2| + \int |M_1-M_2|\cdot|\nabla M_1-\nabla M_2|\cdot|\nabla^3 M_1| \\
\lesssim\ & \|v_1-v_2\|_{L^2}^2\cdot\|\nabla v_2\|_{L^\infty} + \|\nabla M_1-\nabla M_2\|_{L^2}\cdot(\|\nabla M_1\|_{L^\infty}+\|\nabla M_2\|_{L^\infty})\cdot\|\nabla v_1-\nabla v_2\|_{L^2} \\
& + \|\nabla v_1-\nabla v_2\|_{L^2}\cdot\|F_1-F_2\|_{L^2}\cdot(\|F_1\|_{L^\infty}+\|F_2\|_{L^\infty}) + \|F_1-F_2\|_{L^2}\cdot\|v_1-v_2\|_{L^2}\cdot\|\nabla F_2\|_\infty \\
& + \|v_1-v_2\|_{L^2}\cdot\|v_1\|_{L^\infty}\cdot\|\nabla v_1-\nabla v_2\|_{L^2} + \|F_1-F_2\|_{L^2}\cdot\|\nabla F_1-\nabla F_2\|_{L^2}\cdot\|v_1\|_{L^\infty} \\
& + \|F_1-F_2\|_{L^2}^2\cdot\|\nabla v_2\|_{L^\infty} + (\|\nabla v_2\|_{L^\infty}+\|\nabla v_1\|_{L^\infty})\cdot\|\nabla v_1-\nabla v_2\|_{L^2}^2 \\
& + \|v_1-v_2\|_{L^2}\cdot\|\nabla M_1\|_{L^2} + \|\nabla M_2\|_{L^2}\cdot\|\nabla M_1-\nabla M_2\|_{L^2} \\
& + \|\nabla v_1-\nabla v_2\|_{L^2}\cdot[\big(\|\nabla^3 M_1\|_{L^2}+\|\nabla^3 M_2\|_{L^2}\big)\cdot(\|\nabla M_2\|_{L^\infty}+\|\nabla M_1\|_{L^\infty})+\big(\|\nabla^2 M_2\|_{L^4}^2+\|\nabla^2 M_1\|_{L^4}^2\big)] \\
& + \|\nabla v_1-\nabla v_2\|_{L^2}\cdot[\big(\|\nabla^2 F_1\|_{L^2}+\|\nabla^2 F_2\|_{L^2}\big)\cdot(\|F_1\|_{L^\infty}+\|F_2\|_\infty)+\|\nabla F_1\|_{L^\infty}^2+\|\nabla F_2\|_\infty^2] \\
& + \|\nabla v_1-\nabla v_2\|_{L^2}\cdot\|F_1-F_2\|_{L^\infty}(\|\nabla^2 F_1\|_{L^2}+\|\nabla^2 F_2\|_{L^2}) + \|v_1-v_2\|_{L^2}\cdot\|\nabla M_1\|_{L^\infty}\cdot\|\nabla^2 M_1-\nabla^2 M_2\|_{L^2} \\
& + \|v_2\|_{L^\infty}\cdot\|\nabla M_1-\nabla M_2\|_{L^2}\cdot\|\nabla^2 M_1-\nabla^2 M_2\|_{L^2} + \|\nabla M_1-\nabla M_2\|_{L^2}\cdot\|\nabla^3 M_1\|_{L^2}
\end{aligned}
$$

The Sobolev Embeddings $H^s \hookrightarrow L^\infty$ for $s \geq 2$, $H^s \hookrightarrow W^{2,4}$ for $s \geq 3$ and $H^s \hookrightarrow W^{1,\infty}$ for $s \geq 3$ give

$$
\|\nabla v_1\|_{L^\infty} + \|\nabla v_2\|_{L^\infty} \lesssim 1, \qquad \|\nabla M_1\|_{W^{1,\infty}} + \|\nabla M_2\|_{W^{1,\infty}} \lesssim 1
$$

and

$$
\|\nabla M_1\|_{L^4} + \|\nabla M_2\|_{L^4} \lesssim 1, \qquad \|\nabla F_1\|_{W^{1,\infty}} + \|\nabla F_2\|_{W^{1,\infty}} \lesssim 1,
$$

which implies

$$
\begin{aligned}
\frac{d\mathcal{E}}{dt} \lesssim\ & \|v_1-v_2\|_{L^2}^2 + \|\nabla M_1-\nabla M_2\|_{L^2}\cdot\|\nabla v_1-\nabla v_2\|_{L^2} + \|v_1-v_2\|_{L^2} + \|\nabla M_1-\nabla M_2\|_{L^2} \\
& + \|\nabla v_1-\nabla v_2\|_{L^2}\cdot\|F_1-F_2\|_{L^2} + \|F_1-F_2\|_{L^2}\cdot\|v_1-v_2\|_{L^2} + \|\nabla v_1-\nabla v_2\|_{L^2} \\
& + \|v_1-v_2\|_{L^2}\cdot\|\nabla v_1-\nabla v_2\|_{L^2} + \|F_1-F_2\|_{L^2} + \|F_1-F_2\|_{L^2}^2 + \|\nabla v_1-\nabla v_2\|_{L^2}^2 \\
\lesssim\ & \mathcal{E} + \mathcal{E}^{1/2}
\end{aligned}
$$

and ensures the uniqueness.

2.2. **Uniqueness of global existence.** Define

$$\mathcal{E}(t) := \|v_1(t) - v_2(t)\|_{H^1}^2 + \|G_1(t) - G_2(t)\|_{L^2}^2 + \|M_1(t) - M_2(t)\|_{H^1}^2$$

and compute $d\mathcal{E}/dt$. The method is the same as that of **Uniqueness of local existence**. So we omit the proof.

3. Proof for existence

3.1. **Existence of Local solutions.** Construct the following approximate equation

$$\begin{cases} \partial_t v_\varepsilon + v_\varepsilon \cdot \nabla v_\varepsilon + \nabla p + \nabla \cdot (\nabla M_\varepsilon \odot \nabla M_\varepsilon - F_\varepsilon F_\varepsilon^\top) = \nu \Delta v_\varepsilon \\ \nabla \cdot v_\varepsilon = 0 \\ \partial_t F_\varepsilon + v_\varepsilon \cdot \nabla F_\varepsilon = \nabla v_\varepsilon F_\varepsilon \\ \partial_t M_\varepsilon + v_\varepsilon \cdot \nabla M_\varepsilon + M_\varepsilon \times \Delta M_\varepsilon = \varepsilon \left(\Delta M_\varepsilon + |\nabla M_\varepsilon|^2 M_\varepsilon \right) \\ |M_\varepsilon| = 1 \\ (v_\varepsilon, F_\varepsilon, M_\varepsilon)|_{t=0} = (v_0, F_0, M_0)\,. \end{cases}$$

From [2] it follows that the approximate system admits local well-posedness and global well-posedness with small initial data, if we take $H_{\text{ext}} = 0$ in that article.

Lemma 3.1. *([2]) Given an integer $s \geq 2$ and $d \in \{2,3\}$, there exists a positive number T, depending only upon*

$$\mathcal{E}_0 := \|v_0\|_{H^s}^2 + \|F_0\|_{H^s}^2 + \|\nabla M_0\|_{H^s}^2 < \infty$$

such that the approximate system admits a unique solution $(v_\varepsilon, F_\varepsilon, M_\varepsilon)$ satisfying

$$\sup_{t\in[0,T]} \left(\|v_\varepsilon\|_{H^s}^2 + \|F_\varepsilon\|_{H^s}^2 + \|\nabla M_\varepsilon\|_{H^s}^2 \right) + \int_0^T \left(\nu \|\nabla v_\varepsilon\|_{H^s}^2 + \varepsilon \|\Delta M_\varepsilon\|_{H^s}^2 \right) \leq C, \tag{3.1}$$

where C is only dependent of $\mathcal{E}_0$ and T.

3.2. **Proof of local existence.** From (3.1) we know that there exist three functions

$$M \in L^\infty([0,T] \times \mathbb{R}^d, \mathbb{R}^3), \quad F \in L^\infty([0,T], H^s) \quad \textit{and} \quad v \in L^\infty([0,T], H^s)$$

such that, as ε tends to 0,
$M_\varepsilon \longrightarrow M$ weakly-$\star$ in $L^\infty([0,T] \times \mathbb{R}^d, \mathbb{R}^3)$ with $|M| \leq 1$;
$\nabla M_\varepsilon \longrightarrow \nabla M$ weakly-$\star$ in $L^\infty([0,T], H^s)$;
$F_\varepsilon \longrightarrow F$ weakly-$\star$ in $L^\infty([0,T], H^s)$;
$v_\varepsilon \longrightarrow v$ weakly-$\star$ in $L^\infty([0,T], H^s)$,
and

$$\sup_{t\in[0,T]} \left(\|v(t)\|_{H^s}^2 + \|F(t)\|_{H^s}^2 + \|\nabla M(t)\|_{H^s}^2 \right) + \int_0^T \nu \|\nabla v\|_{H^s}^2 \lesssim 1$$

Now we discuss two cases:

Case 1: $s \geq 3$ **or** $s = 2$ **and** $d = 2$

At this case, we say that $H^s \hookrightarrow W^{1,\infty}$ by Sobolev embedding Theorem,

$$\sup_{t\in[0,T]} \|\nabla^2 M(t)\|_{L^\infty}^2 + \int_0^T \nu \|\nabla^2 v\|_{L^\infty}^2 \lesssim 1$$

and

$$\left\{\begin{array}{l}\partial_t v + v\cdot\nabla v + \nabla p + \nabla\cdot(\nabla M\odot\nabla M - FF^\top) = \nu\Delta v\\ \nabla\cdot v = 0\\ \partial_t F + v\cdot\nabla F = \nabla v F\\ \partial_t M + v\cdot\nabla M + M\times\Delta M = 0.\end{array}\right.$$

In the next, we shall prove $|M|\equiv 1$. Indeed, it is easy to get

$$\partial_t\left(|M|^2\right) + v\cdot\nabla\left(|M|^2\right) = 0 \qquad \text{and} \qquad |M(0,\cdot)|^2 = |M_0|^2 \equiv 1$$

which implies our desired result.

Case 2:$s=2$ **and** $d=3$

Note the embedding $H^2(\mathbb{R}^3)\hookrightarrow C^{1/2}(\mathbb{R}^3)$. From (3.1) it follows that for any $\delta\in[0,1/2)$, one are able to get,

$$v_\varepsilon\to v \quad \text{in} \quad C^\delta, \qquad F_\varepsilon\to F \quad \text{in} \quad C^\delta \qquad \text{and} \qquad \nabla M_\varepsilon\to\nabla M \quad \text{in} \quad C^\delta.$$

Taking arbitrary functions $(\Phi,\Psi,\chi)\in C^1([0,T],H^1)$ and $f\in H^1$ to multiply both sides of the approximate equations, we obtain

$$\left\{\begin{array}{l}\int\langle v_\varepsilon(t,\cdot),\Phi(t,\cdot)\rangle - \int\langle v_0,\Phi(0,\cdot)\rangle - \int_0^t\int\langle v_\varepsilon,\partial_t\Phi\rangle - \int_0^t\int\langle v_\varepsilon, v_\varepsilon\cdot\nabla\Phi\rangle + \int_0^t\int\langle\nabla p,\Phi\rangle\\ -\int_0^t\int\langle\nabla M_\varepsilon\odot\nabla M_\varepsilon - F_\varepsilon F_\varepsilon^\top,\nabla\Phi\rangle + \nu\int_0^t\int\langle\nabla v_\varepsilon,\nabla\Phi\rangle = 0\\ \int\langle v_\varepsilon(t,\cdot),\nabla f\rangle = 0\\ \int\langle F_\varepsilon(t,\cdot),\Psi(t,\cdot)\rangle - \int\langle F_0,\Psi(0,\cdot)\rangle - \int_0^t\int\langle F_\varepsilon,\partial_t\Psi\rangle - \int_0^t\int\langle F_\varepsilon, v_\varepsilon\cdot\nabla\Psi\rangle - \int_0^t\int\langle\nabla v_\varepsilon F_\varepsilon,\Psi\rangle = 0\\ \int\langle\Psi(t,\cdot),F_\varepsilon(t,\cdot)\rangle - \int\langle\Psi(0,\cdot),F_0\rangle - \int_0^t\int\langle\partial_t\Psi,F_\varepsilon\rangle - \int_0^t\int\langle v_\varepsilon\cdot\nabla\Psi,F_\varepsilon\rangle - \int_0^t\int\langle\Psi,\nabla v_\varepsilon F_\varepsilon\rangle = 0\\ \int\langle M_\varepsilon(t,\cdot),\chi(t,\cdot)\rangle - \int\langle M_0,\chi(0,\cdot)\rangle - \int_0^t\int\langle M_\varepsilon,\partial_t\chi\rangle - \int_0^t\int\langle M_\varepsilon, v_\varepsilon\cdot\nabla\chi\rangle - \int_0^t\int\langle M_\varepsilon\times\nabla M_\varepsilon,\nabla\chi\rangle\\ +\varepsilon\int_0^t\int\langle\nabla M_\varepsilon,\nabla\chi\rangle - \varepsilon\int_0^t\int|\nabla M_\varepsilon|^2\langle M_\varepsilon,\chi\rangle = 0.\end{array}\right.$$

Combining the weak-$\star$ convergence and strong-C^δ convergence gives

$$\left\{\begin{array}{l}\int\langle v(t,\cdot),\Phi(t,\cdot)\rangle - \int\langle v_0,\Phi(0,\cdot)\rangle - \int_0^t\int\langle v,\partial_t\Phi\rangle - \int_0^t\int\langle v, v\cdot\nabla\Phi\rangle + \int_0^t\int\langle\nabla p,\Phi\rangle\\ -\int_0^t\int\langle\nabla M\odot\nabla M - FF^\top,\nabla\Phi\rangle + \nu\int_0^t\int\langle\nabla v,\nabla\Phi\rangle = 0\\ \int\langle v(t,\cdot),\nabla f\rangle = 0\\ \int\langle F(t,\cdot),\Psi(t,\cdot)\rangle - \int\langle F_0,\Psi(0,\cdot)\rangle - \int_0^t\int\langle F,\partial_t\Psi\rangle - \int_0^t\int\langle F, v\cdot\nabla\Psi\rangle - \int_0^t\int\langle\nabla vF,\Psi\rangle = 0\\ \int\langle\Psi(t,\cdot),F(t,\cdot)\rangle - \int\langle\Psi(0,\cdot),F_0\rangle - \int_0^t\int\langle\partial_t\Psi,F\rangle - \int_0^t\int\langle v\cdot\nabla\Psi,F\rangle - \int_0^t\int\langle\Psi,\nabla vF\rangle = 0\\ \int\langle M(t,\cdot),\chi(t,\cdot)\rangle - \int\langle M_0,\chi(0,\cdot)\rangle - \int_0^t\int\langle M,\partial_t\chi\rangle - \int_0^t\int\langle M, v\cdot\nabla\chi\rangle - \int_0^t\int\langle M\times\nabla M,\nabla\chi\rangle = 0\\ |M| = 1 \qquad a.e.\end{array}\right.$$

Now we are going to prove $|M|=1$ almost everywhere. Otherwise, there exists a number $\eta\in(0,1)$ such that the measure of the set $S:=\{(t,x)\in\mathbb{R}\times\mathbb{R}^3 : |M|\le\eta\}$ is positive. Without loss of generality, we assume that

$$\mathcal{B}_\epsilon := [0,\epsilon)\times\mathbb{B}(0,\epsilon)\subset S,$$

where $\mathbb{B}(0,\epsilon)$ is the open ball in $\mathbb{R}^3$ with radius ϵ centerring at the origin.

Let $\chi := \phi M$ with $\phi\in C_c^\infty(\mathcal{B}_\epsilon)$ being non-negative and always being 1 in $\mathcal{B}_{\epsilon/2}$. It is not difficult to get

$$\int_{\mathbb{B}(0,\epsilon/2)}\phi\left(|M(t,\cdot)|^2 - 1\right) - \frac{1}{2}\int_0^t\int_{\mathbb{B}(0,\epsilon/2)}(|M|^2-1)v\cdot\nabla\phi = 0$$

which is equivalent to

$$\int_{\mathbb{B}(0,\epsilon/2)} \left(|M(t,\cdot)|^2 - 1\right) = 0$$

for all $t \in [0, \epsilon/2)$. However,

$$\int_{\mathbb{B}(0,\epsilon/2)} \left(|M(t,\cdot)|^2 - 1\right) \le (\eta^2 - 1)|\mathbb{B}(0,\epsilon/2)| < 0$$

which means a contradiction.

3.3. **Existence of global solutions.**

Lemma 3.2. *([2]) Under the assumptions of Lemma 3.1, the approximate system admits a unique solution $(v_\varepsilon, F_\varepsilon, M_\varepsilon)$ constructed in Lemma 3.1 satisfying*

$$\begin{aligned}&\|v_\varepsilon(t)\|_{H^s}^2 + \|G_\varepsilon(t)\|_{H^s}^2 + \|\nabla M_\varepsilon(t)\|_{H^s}^2 + \|\partial_t v_\varepsilon(t)\|_{H^{s-2}}^2 + \|\partial_t G_\varepsilon(t)\|_{H^{s-2}}^2 \\ &+ \int_0^t \left(\nu\|\nabla v_\varepsilon(\tau)\|_{H^s}^2 + \varepsilon\|\Delta M_\varepsilon(\tau)\|_{H^s}^2\right) d\tau \lesssim \epsilon_0\end{aligned} \tag{3.2}$$

for all $t \ge 0$ with $G_\varepsilon := F_\varepsilon^{-1} - I$, if we assume furhter that $G_0 := F_0^{-1} - I$ satisfies

$$\partial_i G_0^{jk} = \partial_k G_0^{ji} \qquad\qquad \forall i,j,k = 1,2,\cdots,d$$

and that

$$detF_0 = 1, \qquad \|v_0\|_{H^s}^2 + \|F_0 - I\|_{H^s}^2 + \|\nabla M_0\|_{H^s}^2 \le \epsilon_0$$

for some small positive number ϵ_0.

3.4. **Proof of global existence.** Defining

$$(A \sharp \nabla B)_k := \sum_{j=1}^3 \sum_{i=1}^3 A_{ji}\partial_j B_{ik}$$

tells us that the approximate system is equivalent to

$$\begin{cases} \partial_t v_\varepsilon + v_\varepsilon \cdot \nabla v_\varepsilon + \nabla p + \nabla\cdot(\nabla M_\varepsilon \odot \nabla M_\varepsilon) + (F_\varepsilon \sharp \nabla G_\varepsilon^\top) F_\varepsilon F_\varepsilon^\top \\ +[(F_\varepsilon F_\varepsilon^\top)\sharp \nabla G_\varepsilon^\top]F_\varepsilon^\top = \nu\Delta v_\varepsilon \\ \nabla\cdot v_\varepsilon = 0 \\ \partial_t G_\varepsilon + v_\varepsilon\cdot\nabla G_\varepsilon + (G_\varepsilon + I)\nabla v_\varepsilon = 0 \\ \partial_t M_\varepsilon + v_\varepsilon\cdot\nabla M_\varepsilon + M_\varepsilon \times \Delta M_\varepsilon = \varepsilon\left(\Delta M_\varepsilon + |\nabla M_\varepsilon|^2 M_\varepsilon\right) \\ |M_\varepsilon| = 1. \end{cases} \tag{3.3}$$

From (3.2) we know that there exist three functions

$$M \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}^3), \quad G \in L^\infty(\mathbb{R}^+, H^s) \quad and \quad v \in L^\infty(\mathbb{R}^+, H^s)$$

such that, as ε tends to 0,
$M_\varepsilon \longrightarrow M$ weakly-$\star$ in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}^3)$ with $|M| \le 1$;
$\nabla M_\varepsilon \longrightarrow \nabla M$ weakly-$\star$ in $L^\infty(\mathbb{R}^+, H^s)$;
$G_\varepsilon \longrightarrow G$ weakly-$\star$ in $L^\infty(\mathbb{R}^+, H^s)$;
$\partial_t G_\varepsilon \longrightarrow \partial_t G$ weakly-$\star$ in $L^\infty(\mathbb{R}^+, H^{s-2})$;
$v_\varepsilon \longrightarrow v$ weakly-$\star$ in $L^\infty(\mathbb{R}^+, H^s)$;

$\partial_t v_\varepsilon \longrightarrow \partial_t v$ weakly-$\star$ in $L^\infty(\mathbb{R}^+, H^{s-2})$,
and

$$\|v(t)\|_{H^s}^2 + \|G(t)\|_{H^s}^2 + \|\nabla M(t)\|_{H^s}^2 + \|\partial_t v(t)\|_{H^{s-2}}^2 + \|\partial_t G(t)\|_{H^{s-2}}^2 + \int_0^t \nu\|\nabla v(\tau)\|_{H^s}^2 d\tau \lesssim \epsilon_0$$

for all $t \geq 0$. Now we study two cases:

Case 1: $s \geq 3$ **or** $s = 2$ **and** $d = 2$

At this case, Sobolev Embedding Theorem implies $H^s \hookrightarrow W^{1,\infty}$ and

$$\|\nabla^2 M(t)\|_{L^\infty}^2 + \int_0^t \nu\|\nabla^2 v\|_{L^\infty}^2 \lesssim \epsilon_0.$$

From the approximate system it follows that

$$\begin{aligned}
\|\partial_t F_\varepsilon\|_{H^{s-2}} &\leq \|v_\varepsilon \cdot \nabla F_\varepsilon\|_{H^{s-2}} + \|\nabla v_\varepsilon (F_\varepsilon - I)\|_{H^{s-2}} + \|\nabla v_\varepsilon\|_{H^{s-2}} \\
&\leq \|v_\varepsilon\|_{L^\infty}\|\nabla F_\varepsilon\|_{H^{s-2}} + \|\nabla v_\varepsilon\|_{L^\infty}\|F_\varepsilon - I\|_{H^{s-2}} + \|\nabla v_\varepsilon\|_{H^{s-2}} \\
&\leq \|v_\varepsilon\|_{W^{1,\infty}}\|F_\varepsilon - I\|_{H^{s-1}} + \|\nabla v_\varepsilon\|_{H^{s-2}} \lesssim (\|v_\varepsilon\|_{H^s} + 1)\|F_\varepsilon - I\|_{H^{s-1}} \\
&\sim (\|v_\varepsilon\|_{H^s} + 1)\|G_\varepsilon\|_{H^{s-1}} \lesssim \epsilon_0^{1/2},
\end{aligned}$$

where we have used (3.3) of [3]. Moreover, (3.3) of [3] also tells us that we can find a function F such that $F_\varepsilon - I \longrightarrow F - I$ weakly-$\star$ in $L^\infty(\mathbb{R}^+, H^s)$, which means $\partial_t F_\varepsilon \longrightarrow \partial_t F$ weakly-$\star$ in $L^\infty(\mathbb{R}^+, H^{s-2})$, $F_\varepsilon \longrightarrow F$ weakly-$\star$ in $L^\infty(\mathbb{R}^+, W^{1,\infty})$, $\|F_\varepsilon\|_{L^\infty(\mathbb{R}^+, W^{1,\infty})} \lesssim 1 + \epsilon_0^{1/2}$ and $\|F - I\|_{L^\infty(\mathbb{R}^+, W^{1,\infty})} \lesssim \epsilon_0^{1/2}$. Now we say $F(G + I) = I$, since $F_\varepsilon(G_\varepsilon + I) = I$.

Appling the same method of **Case 1** of **Local existence** gives

$$\begin{cases}
\partial_t v + v\cdot\nabla v + \nabla p + \nabla\cdot(\nabla M \odot \nabla M) + (F \sharp \nabla G^\top) FF^\top + [(FF^\top) \sharp \nabla G^\top]F^\top = \nu\Delta v \\
\nabla \cdot v = 0 \\
\partial_t G + v\cdot\nabla G + (G + I)\nabla v = 0 \\
\partial_t M + v\cdot\nabla M + M \times \Delta M = 0 \\
|M| = 1,
\end{cases}$$

which is equivalent to (1.2).

Case 2: $s = 2$ **and** $d = 3$

The approach is similar to that of **Case 2** of **Local existence**. Hence we omit it.

This completes all the proof.

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